國立嘉義大學九十四學年度
應用數學系碩士班招生考試試題

科目：線性代數

說明：本考試試題為計算、證明題，請標明題號，同時將過程作答在「答案卷」上。（1~2 題每題 20 分，3~6 題每題 15 分，共 100 分）

1. Let $A, B \in M_{n\times n}(F)$. We say that $B \sim A$ if there exists an invertible Q such that $B = Q^{-1}AQ$.

   (a). Prove that "~" is an equivalence relation. （5 分）

   (b). Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 4 & 1 \\ 8 & 6 & -1 \end{pmatrix}$. Is $A \sim B$? Explain your answer. （5 分）

   (c). Assume $A \sim B$. Is $tr(A) = tr(B)$? (Prove it or give me a counterexample) （10 分）

2. For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

   (a). $V = \mathrm{R}^4$, $T(a,b,c,d) = (a+b, b-c, a+c, a+d)$, and $z = e_1$. （5 分）

   (b). $V = P_3(\mathrm{R})$. $T(f) = f''$, and $z = x^3$. （5 分）

   (c). $V = M_{2\times 2}(\mathrm{R})$, $T(A) = A^t$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. （5 分）

   (d). $V = M_{2\times 2}(\mathrm{R})$, $T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. （5 分）

3. (a). State the definition of vector subspace over a field F. （5 分）

   (b). Let $V = \{(x,y) \in \mathrm{R}^2 \mid 2x + y = k\}$ for some $k \in \mathrm{R}$. Find all k such that V is an one dimensional vector subspace of R^2 over R. Please explain your answer. （10 分）

4. Let C be the classical adjoint of $A \in M_{n\times n}(F)$. Prove the following.

   (a). $\det(C) = [\det(A)]^{n-1}$. （5 分）

   (b). C^t is the classical adjoint of A^t. （5 分）

   (c). If A is an invertible upper triangular matrix, then C and A^{-1} are both upper triangular matrices. （5 分）

5. Let $A = \begin{pmatrix} -6 & -6 & 5 \\ 4 & 2 & -1 \\ -6 & -4 & 3 \end{pmatrix}$.

   (a). Find the characteristic polynomial of A. （2 分）

   (b). Find all eigenvalues and eigenvectors of A. （5 分）

   (c). Let T be a linear operator on a vector space V and let λ be the eigenvalue of T. Prove that: $v \in V$ is an eigenvector of T corresponding to λ iff $v \neq 0$ and $v \in N(T - \lambda I)$ （8 分）

6. Let T and U be positive definite operators on an inner product space V. Prove the following.

   (a). $T + U$ is positive definite. （5 分）

   (b). If $c > 0$, then cT is positive definite. （5 分）

   (c). T^{-1} is positive definite. （5 分）