

國立嘉義大學九十七學年度
應用數學系碩士班（甲組）招生考試試題

科目：高等微積分

說明：本考試試題為計算、證明題，請標明題號，同時將過程作答在「答案卷」上。(1~3 題每題 20 分，4~5 題每題 15 分，第 6 題 10 分，共 100 分)

1. (a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be an additive function, i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at $x=0$, show that f is continuous everywhere on \mathbb{R} . (10%)

(b) Let $g: \mathbb{R} \rightarrow \mathbb{R}$, where $g(x) = \begin{cases} -2x & \text{if } x \text{ is a positive rational number,} \\ 2x & \text{if } x \text{ is a negative rational number,} \\ 0 & \text{otherwise.} \end{cases}$

Determine the continuity of g and justify your answer. (10%)

2. Determine whether the following statements true or false and justify your answers. (Hint: Prove your answer or find a counter example.)

(a) Let $\{u_i\}$ be a sequence of real numbers. If $\sum_{i=1}^{\infty} u_i$ converges, then $\sum_{i=1}^{\infty} (u_i)^3$ converges, too. (5%)

(b) Let $[a, b] \times [c, d] \equiv \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b], y \in [c, d]\}$ and $f: [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a real value function. If $\int_a^b \left(\int_c^d f(x, y) dy \right) dx$ and $\int_c^d \left(\int_a^b f(x, y) dx \right) dy$ are exist, then $\int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$. (5%)

(c) Let A be a compact subset of \mathbb{R}^n . If the function f is continuous on A , then $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is uniformly continuous on A . (5%)

(d) $f(x) = \sum_{n=1}^{\infty} \left(\frac{\sin nx}{n^2} \right) x^3$ is continuous on \mathbb{R} . (5%)

3. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ real-valued functions, where

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x, y) = \begin{cases} \frac{x^4 + y^4}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Determine whether f and g are differentiable at $(0, 0)$ and justify your answer. (20%)

4. Prove the following statements:

(a) If $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series, then the series $\sum_{n=0}^{\infty} a_n \cos(nx)$ is absolutely and uniformly convergent on the interval $[-\pi, \pi]$. (8%)

(b) If $\{b_n\}$ is a decreasing sequence of positive numbers and the series $\sum_{n=1}^{\infty} b_n \cos(nx)$ is uniformly convergent on the interval $[-\pi, \pi]$, then $\lim_{n \rightarrow \infty} n b_n = 0$. (7%)

5. Show that every bounded monotone real value function is integrable. (15%)

6. Find the extreme values of the function $f(x, y, z) = 4xy + 2xz + 2yz$ subject to the constraint $xyz = 16$. (10%)