國立嘉義大學九十六學年度
應用數學系碩士班招生考試（甲組）試題

科目：高等微積分

說明：本試題為計算、證明題，請標明題號，同時將過程寫在「答案卷」上。

1. (a) Let h be a real-valued function. Let h and its derivative h' be defined on the interval (a, b) and with values in $\Re$. Let $\left|h'(x)\right| \le M$ for all $a \in (a, b)$, where M is a positive real number. Show that h is uniformly continuous on (a, b). （10 分）

   (b) Let $q(x) = \dfrac{1}{x}$, $x \in (0, 1)$. Show that q is not uniformly continuous on $(0,1)$. （10 分）

2. Suppose that $a_n, b_n > 0, \forall\ n \in \aleph$.

   (a) Show that if $\dfrac{a_{n+1}}{a_n} \le \dfrac{b_{n+1}}{b_n}, \forall\ n \in \aleph$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. （10 分）

   (b) Show that if $p > 1$, $\dfrac{a_{n+1}}{a_n} \le 1 - \dfrac{p}{n+1}, \forall\ n \in \aleph$, then $\sum_{n=1}^{\infty} a_n$ converges. （10 分）

3. (a) Let $\{a_n\}$ be a sequence of real numbers. Show that $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$. （8 分）

   (b) Let f_n be defined on $\Re$ by $f_n(x) = \dfrac{n^3 x^3}{1 + n^4 x^4}$, $n = 1, 2, 3, \Lambda$. Show that $\{f_n\}$ converges on $\Re$. （7 分）

4. Let $D = \left\{(x, t) \middle|\ a \le x \le b, c \le t \le d\ \right\}$ be a rectangle in $\Re \times \Re$. Let $g : D \to \Re$ be a function. Let g and its partial derivative g_t be continuous on D. Show that

$$\frac{d}{dt}\int_a^b g(x, t)\, dx = \int_a^b g_t(x, t)\, dx, \quad c \le t \le d. \quad (15\text{ 分})$$

5. (a) Find a counter example for the statement: If A, B are closed subsets of $\Re^n$, so is $A + B$, where $A + B = \left\{a + b \middle|\ a \in A, b \in B\right\}$. （5 分）

   (b) Suppose that A is a closed subset of $\Re^n$ and B is a compact subset of $\Re^n$. Show that $A + B$ is closed. （10 分）

6. Determine whether each of the following sequences of functions converges uniformly on $\Re$ and justify your answer.

   (a) $f_n : \Re \to \Re$, $f_n(x) = \begin{cases} 1 & \text{if } x \in (-n, n) \\ 0 & \text{otherwise} \end{cases}$, $\forall\ n \in \aleph$. （5 分）

   (b) $g_n : \Re \to \Re$, $g_n(x) = \begin{cases} n & \text{if } x \in \left(0, 1/n^2\right) \\ 0 & \text{otherwise} \end{cases}$, $\forall\ n \in \aleph$. （5 分）

   (c) $h_n : \Re \to \Re$, $h_n(x) = \begin{cases} x/n^2 & \text{if } x \in [0, n] \\ 1/n & \text{if } x \in (n, \infty) \\ 0 & \text{if } x \in (-\infty, 0) \end{cases}$, $\forall\ n \in \aleph$. （5 分）